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Generalization of Alternative Rings, I

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INTRODUCTION

In this paper we shall study rings R which satisfy for the most part the following three identities:

- (i) $g(w, x, y, z) = (wx, y, z) + (w, x, (y, z))$
 $\quad - w(x, y, z) - (w, y, z)x = 0,$
- (ii) $h(w, x, y, z) = ((w, x), y, z) + (w, x, yz)$
 $\quad - y(w, x, z) - (w, x, y)z = 0,$
- (iii) $(x, x, x) = 0,$

where we define $(a, b, c) = (ab)c - a(bc)$, and $(a, b) = ab - ba$. It is well known that (i) is true in a right alternative ring of characteristic different from two. But being an identity of degree four it is of course weaker than the right alternative identity. Then (ii) is the counterpart of (i), and a consequence of the left alternative identity. A number of generalizations of the alternative identities have been considered, some of which are listed in the reference section. Some of these have taken on the form of identities which are shared by commutative and hence Jordan rings, while others are shared by Lie rings. The object of this generalization is to ascertain whether there exist interesting examples of rings like the Cayley numbers, hence our choice of (i) and (ii). Throughout our rings will be assumed to have characteristic different from two and three, meaning in this case that there exist no elements of additive order two and three.

The main results are the following. A necessary and sufficient condition for a ring of characteristic different from two and three, satisfying (i) and (ii) to be alternative is that whenever there exist elements a, b, c which are contained in a subring which can be generated by two elements and $(a, b, c)^2 = 0$, then $(a, b, c) = 0$. So therefore a ring without nilpotent

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elements other than zero, satisfying (i) and (ii) must be alternative. A ring satisfying (i)–(iii) that contains an idempotent $e \neq 1$, and which has no proper ideal must be alternative. This rules out completely the existence of any new rings like the Cayley numbers, even though the identities are substantially weaker than the alternative identities.

PART I

Initially we shall consider a ring R which satisfies (i)–(iii) and establish the necessary and sufficient condition mentioned in the Introduction. This will be followed by calculations that show we can dispense with (iii) and still obtain the desired result. First

$$\begin{aligned} 0 &= g(y, x, x, x) = (yx, x, x) + (y, x, (x, x)) - y(x, x, x) - (y, x, x)x \\ &= (yx, x, x) - (y, x, x)x, \end{aligned}$$

as a result of (iii). Thus

$$(yx, x, x) = (y, x, x)x. \quad (1)$$

We define $a \circ b = ab + ba$. Then

$$0 = g(y, y, x, x) = (y^2, x, x) + (y, y, (x, x)) - y(y, x, x) - (y, x, x)y,$$

so that $(y^2, x, x) = y \circ (y, x, x)$. Linearizing the last identity, it becomes clear that $(y \circ z, x, x) = y \circ (z, x, x) + z \circ (y, x, x)$. If we put $z = x$, then $(y \circ x, x, x) = x \circ (y, x, x)$, because of (iii). Comparing the last equation with (1), it must be that

$$(xy, x, x) = x(y, x, x). \quad (2)$$

Throughout the paper it is useful to go to the anti-isomorphic copy of R . By this we mean the additive group of R together with a new multiplication defined by $x * y = yx$. Clearly the same identities, whether (i) and (ii), or (i)–(iii) will also hold in the new ring. Since (1) and (2) are identities in the anti-isomorphic copy of R , it must be that

$$(x, x, xy) = x(x, x, y), \quad (3)$$

and

$$(x, x, yx) = (x, x, y)x. \quad (4)$$

A linearization of (iii) shows that $(x, x, z) + (x, z, x) + (z, x, x) = 0$. We may substitute $z = xy$, to obtain $(x, xy, x) = -(xy, x, x) - (x, x, xy) =$

$-x(y, x, x) - x(x, x, y) = x(x, y, x)$, using (2), (3) and another linearization of (iii). So

$$(x, xy, x) = x(x, y, x). \quad (5)$$

By going to the anti-isomorphic copy of R , (5) implies

$$(x, yx, x) = (x, y, x)x. \quad (6)$$

In an arbitrary ring one may verify the Teichmüller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z.$$

We shall have many occasions to use this identity. Whenever we wish to apply it to the elements w, x, y, z of the ring we shall write $0 = f(w, x, y, z)$. Then

$$\begin{aligned} 0 &= f(x, y, x, x) = (xy, x, x) - (x, yx, x) + (x, y, x^2) \\ &\quad - x(y, x, x) - (x, y, x)x \\ &= x(y, x, x) - (x, y, x)x + (x, y, x^2) - x(y, x, x) - (x, y, x)x, \end{aligned}$$

using (2) and (6). After cancelling two terms we see that

$$(x, y, x^2) = 2(x, y, x)x. \quad (7)$$

By going to the anti-isomorphic copy of R it follows from (7) that

$$(x^2, y, x) = 2x(x, y, x). \quad (8)$$

But

$$\begin{aligned} h(x, y, x, x) &= ((x, y), x, x) + (x, y, x^2) - x \circ (x, y, x) \\ &= ((x, y), x, x) + 2(x, y, x)x - x \circ (x, y, x), \end{aligned}$$

using (7). Hence cancellation shows that

$$((x, y), x, x) = x(x, y, x) - (x, y, x)x = (x, (x, y, x)).$$

But use of (1) and (2) indicates that

$$\begin{aligned} ((x, y), x, x) &= (xy, x, x) - (yx, x, x) \\ &= x(y, x, x) - (y, x, x)x = (x, (y, x, x)). \end{aligned}$$

Hence

$$(x, (y, x, x)) = (x, (x, y, x)). \quad (9)$$

By going to the anti-isomorphic copy of R it follows from (9) that

$$(x, (x, x, y)) = (x, (x, y, x)). \quad (10)$$

A linearization of (iii) shows that $(x, x, y) + (x, y, x) + (y, x, x) = 0$. Then combining $0 = (x, (x, x, y) + (x, y, x) + (y, x, x))$, with (9) and (10) it follows that $3(x, (x, y, x)) = 0$. Using characteristic different from three it becomes clear that $(x, (x, y, x)) = 0$. Combining this with (9) and (10) we get

$$(x, (x, y, x)) = 0 = (x, (y, x, x)) = (x, (x, x, y)). \quad (11)$$

Define $u = (x, y, x)$. Then (11) implies $(x, u) = 0$. In the last identity replace y by xy . Then $(x, (x, xy, x)) = 0$. But $(x, xy, x) = xu$, because of (5), so that $(x, xu) = 0$. Hence $0 = (x, xu) = x(xu) - (xu)x = x(ux) - (xu)x$, using $xu = ux$. But $x(ux) - (xu)x = -(x, u, x)$, and thus $(x, u, x) = 0$, or

$$(x, (x, y, x), x) = 0. \quad (12)$$

In the course of proving (1) we observed that $(y^2, x, x) = y \circ (y, x, x)$. By going to the anti-isomorphic copy of R it follows that $(x, x, y^2) = y \circ (x, x, y)$. By linearizing (iii) we see that $(x, x, y^2) + (x, y^2, x) + (y^2, x, x) = 0$. Then

$$\begin{aligned} (x, y^2, x) &= -(y^2, x, x) - (x, x, y^2) \\ &= -y \circ (y, x, x) - y \circ (x, x, y) = y \circ (x, y, x). \end{aligned}$$

Substituting y^2 for y in (12) then $0 = (x, (x, y^2, x), x) = (x, y \circ u, x)$. But $(x, y^2, x) = y \circ (x, y, x)$ may be linearized to show that $(x, y \circ z, x) = y \circ (x, z, x) + z \circ (x, y, x)$. Let $z = u$ in the last identity. Then

$$0 = (x, y \circ u, x) = y \circ (x, u, x) + u \circ (x, y, x).$$

But $(x, u, x) = 0$, as a consequence of (12), while $u = (x, y, x)$, so that $u \circ (x, y, x) = u \circ u = 2u^2$. Thus $0 = 2u^2$. Using characteristic different from two we see that $u^2 = 0$. Then the condition we are using in our hypothesis implies that $u = 0$. Thus $(x, y, x) = 0$, or the flexible identity holds. We have proved

LEMMA 1. R is flexible.

Now $0 = g(x, x, y, z) = (x^2, y, z) + (x, x, (y, z)) - x \circ (x, y, z)$, while $0 = h(z, y, x, x) = ((z, y), x, x) + (z, y, x^2) - x \circ (z, y, x)$. Combining the last two identities and using the linearization of the flexible identity it follows that $2(x, x, (y, z)) = 0$. Using characteristic different from two then $(x, x, (y, z)) = 0$. But now $0 = g(x, x, y, z)$ becomes $(x^2, y, z) = x \circ (x, y, z)$, while $0 = h(z, y, x, x)$ becomes $(z, y, x^2) = x \circ (z, y, x)$. But now we can

use the main result of [3] to prove that R must be alternative. Of course in an alternative ring the condition is automatically satisfied because of Artin's theorem. For the remainder of Part I we shall consider a ring R which satisfies identities (i) and (ii) but not necessarily (iii), and establish that the condition suffices to prove (iii), so that R must be alternative. From $0 = g(x, x, x, x) = (x^2, x, x) - x \circ (x, x, x)$, we obtain

$$(x^2, x, x) = x \circ (x, x, x). \quad (13)$$

Then $0 = h(x, x, x, x) = (x, x, x^2) - x \circ (x, x, x)$, we obtain

$$(x, x, x^2) = x \circ (x, x, x). \quad (14)$$

But

$$0 = f(x, x, x, x) = (x^2, x, x) - (x, x^2, x) + (x, x, x^2) - x(x, x, x) - (x, x, x)x,$$

implies, using (13) and (14), that

$$(x, x^2, x) = x \circ (x, x, x). \quad (15)$$

Let $t = (x, x, x)$. Then $(t, x, x) = (x^2 \cdot x, x, x) - (x \cdot x^2, x, x)$. But

$$0 = g(x^2, x, x, x) = (x^2 \cdot x, x, x) - x^2 t - (x^2, x, x)x,$$

so that $(x^2 \cdot x, x, x) = x^2 t + (x^2, x, x)x = x^2 t + (x \circ t)x$, using (13). Also $0 = g(x, x^2, x, x) = (x \cdot x^2, x, x) - x(x^2, x, x) - t \cdot x^2$, so that $(x \cdot x^2, x, x) = x(x \circ t) + t \cdot x^2$. Thus $(t, x, x) = x^2 t + (xt)x + (tx)x - tx^2 - x(xt) - x(tx) = (x, x, t) + (x, t, x) + (t, x, x)$, so

$$(x, x, t) = -(x, t, x). \quad (16)$$

Similarly, by going to the anti-isomorphic copy of R , (16) becomes

$$(t, x, x) = -(x, t, x) = (x, x, t). \quad (17)$$

But $0 = g(x, x, x, x^2) = (x^2, x, x^2) - (x, x, t) - x \circ (x, x, x^2)$, while

$$0 = h(x^2, x, x, x) = (t, x, x) + (x^2, x, x^2) - x \circ (x^2, x, x).$$

Since $(x^2, x, x) = x \circ t = (x, x, x^2)$, using (13) and (14), we may compare the previous two equations and obtain $(t, x, x) = -(x, x, t)$. But then use of (17) and characteristic different from two leads to

$$(x, t, x) = 0 = (t, x, x) = (x, x, t). \quad (18)$$

Linearizing (18) we find that

$$((y, x, x) + (x, y, x) + (x, x, y), x, x) + (t, y, x) + (t, x, y) = 0.$$

In the last equation replace y by x^2 . Then

$$3(t \circ x, x, x) + (t, x^2, x) + (t, x, x^2) = 0,$$

using (13)–(15). Since $(y^2, x, x) = y \circ (y, x, x)$, we linearize to obtain $(y \circ z, x, x) = y \circ (z, x, x) + z \circ (y, x, x)$. Replacing z by t in the last identity it becomes clear that $(t \circ y, x, x) = t \circ (y, x, x) + y \circ (t, x, x) = t \circ (y, x, x)$, using (18). Now let $y = x$ in the last identity. Then $(t \circ x, x, x) = 2t^2$. Thus

$$6t^2 + (t, x^2, x) + (t, x, x^2) = 0. \quad (19)$$

Then $0 = f(t, x, x, x) = (tx, x, x) - (t, x^2, x) + (t, x, x^2) - t^2 - (t, x, x)x = (tx, x, x) - (t, x^2, x) + (t, x, x^2) - t^2$, using (18). But $0 = g(t, x, x, x) = (tx, x, x) - t^2 - (t, x, x)x = (tx, x, x) - t^2$, using (18). Hence $(tx, x, x) = t^2$. Comparing this with the identity following (19) we see that $(t, x^2, x) = (t, x, x^2)$, so

$$(t, x^2, x) = (t, x, x^2). \quad (20)$$

Next $(x, x, x^2) = x \circ t$. Linearizing and replacing x by $x + t$ and $x - t$ and comparing, we are led to $(x, x, t \circ x) + (x, t, x^2) + (t, x, x^2) = 2t^2$, in the light of (18). But $(x, x, y^2) = y \circ (x, x, y)$, so that $(x, x, t \circ x) = 2t^2$, using (18). Thus

$$(t, x, x^2) = -(x, t, x^2). \quad (21)$$

Now

$$\begin{aligned} 0 &= h(x, t, x, x) = (xt - tx, x, x) + (x, t, x^2) - x \circ (x, t, x) \\ &= (xt - tx, x, x) + (x, t, x^2), \end{aligned}$$

using (18). On the other hand, $0 = g(x, t, x, x) = (xt, x, x) - x(t, x, x) - t^2 = (xt, x, x) - t^2$, using (18). Thus $(xt, x, x) = t^2$. Also $0 = g(t, x, x, x) = (tx, x, x) - t^2 - (t, x, x)x = (tx, x, x) - t^2$, using (18). This implies $(tx, x, x) = t^2$. But then $(xt - tx, x, x) = t^2 - t^2 = 0$. Comparing this with the equation following (21), we see that $(x, t, x^2) = 0$. At this point (21), (20) and (19) imply that $6t^2 = 0$, so that $t^2 = 0$. Using the condition, then $t = 0$. Since $t = (x, x, x)$, we have established identity (iii) in R . Now our previous work may be invoked to show that R is alternative. This proves

THEOREM 1. *A necessary and sufficient condition for a ring of characteristic different from two and three, satisfying identities (i) and (ii) to be alternative is that whenever there exist elements a, b, c which are contained in a subring which can be generated by two elements and $(a, b, c)^2 = 0$, then $(a, b, c) = 0$.*

PART II

In this part we deal with rings R that satisfy identities (i)–(iii), possess an idempotent $e \neq 1$, have characteristic different from two and three and are simple. Initially we shall add an extra assumption, namely that $(e, e, R) = 0 = (e, R, e) = (R, e, e)$. In the latter part we shall remove this from the hypothesis. Then we can prove that R must be alternative.

It is well known that because of the associativity conditions on e that R has a Peirce decomposition. This means R can be written as a direct sum $R = R_{00} + R_{01} + R_{10} + R_{11}$, where if $x_{ij} \in R_{ij}$ and $i, j = 0$ or 1 , then $ex_{ij} = ix_{ij}$, $x_{ij}e = jx_{ij}$. It is convenient to observe what happens to the Peirce decomposition when we pass to the anti-isomorphic copy of R . Of course, e remains the same, but R_{ij} is changed to R_{ji} , while multiplication is reversed. A second trick that turns out convenient is changing subscripts, without reversing multiplication. Formally this is possible only when R has an identity element 1 and then it means doing the Peirce decomposition relative to the idempotent $1 - e$ instead of e . However in practice one can repeat the computation even without the element 1 . Wherever such work is required, the work will be left to the reader.

First

$$\begin{aligned} 0 &= g(e, x_{10}, e, y_{11}) = (ex_{10}, e, y_{11}) + (e, x_{10}, ey_{11} - y_{11}e) \\ &\quad - e(x_{10}, e, y_{11}) - (e, e, y_{11})x_{10}. \end{aligned}$$

The second and fourth terms of the last equation vanish however, leaving $(x_{10}, e, y_{11}) = e(x_{10}, e, y_{11})$. But

$$\begin{aligned} 0 &= h(x_{10}, e, e, y_{11}) = (x_{10}e - ex_{10}, e, y_{11}) \\ &\quad + (x_{10}, e, ey_{11}) - e(x_{10}, e, y_{11}) - (x_{10}, e, e)y_{11}, \end{aligned}$$

and so $-(x_{10}, e, y_{11}) + (x_{10}, e, y_{11}) - e(x_{10}, e, y_{11}) = 0$. Since the first two terms of the last equation cancel, we are left with $e(x_{10}, e, y_{11}) = 0$. Comparing this with our previous identity, we are led to $(x_{10}, e, y_{11}) = 0$. Hence $x_{10}y_{11} = 0$, so that $R_{10}R_{11} = 0$. By going to the anti-isomorphic copy of R we are led to $R_{11}R_{01} = 0$. Then by reversing subscripts we obtain $R_{01}R_{00} = 0$, and $R_{00}R_{10} = 0$. Thus

$$R_{10}R_{11} = 0 = R_{11}R_{01} = R_{01}R_{00} = R_{00}R_{10}. \quad (22)$$

Now

$$\begin{aligned} 0 &= h(x_{11}, e, e, y_{00}) = (x_{11}e - ex_{11}, e, y_{00}) + (x_{11}, e, ey_{00}) \\ &\quad - e(x_{11}, e, y_{00}) - (x_{11}, e, e)y_{00}. \end{aligned}$$

All but the third term vanish and we obtain $-e(x_{11}y_{00}) = 0$. Also

$$\begin{aligned} 0 &= h(x_{11}, e, y_{00}, e) = (x_{11}e - ex_{11}, y_{00}, e) \\ &\quad + (x_{11}, e, y_{00}e) - y_{00}(x_{11}, e, e) - (x_{11}, e, y_{00})e. \end{aligned}$$

All but the last term vanish and we are left with $-(x_{11}y_{00})e = 0$. Thus we have shown $x_{11}y_{00} \in R_{00}$. But then

$$0 = g(e, e, x_{11}, y_{00}) = (e, x_{11}, y_{00}) + (e, e, x_{11}y_{00} - y_{00}x_{11}) - e \circ (e, x_{11}, y_{00}).$$

Now the second term vanishes by hypothesis. Since $(e, x_{11}, y_{00}) = x_{11}y_{00} - e(x_{11}y_{00}) = x_{11}y_{00} \in R_{00}$, we also have $e \circ (e, x_{11}, y_{00}) = 0$. Thus only the first term survives, so that $0 = (e, x_{11}, y_{00}) = x_{11}y_{00}$. By going to the anti-isomorphic copy of R we find that

$$R_{11}R_{00} = 0 = R_{00}R_{11}. \quad (23)$$

Now

$$\begin{aligned} 0 &= h(y_{01}, e, e, x_{01}) = (y_{01}e - ey_{01}, e, x_{01}) \\ &\quad + (y_{01}, e, ex_{01}) - e(y_{01}, e, x_{01}) - (y_{01}, e, e)x_{01}. \end{aligned}$$

Clearly the second and fourth terms vanish. What remains is

$$(y_{01}, e, x_{01}) - e(y_{01}, e, x_{01}) = 0,$$

so that $y_{01}x_{01} = e(y_{01}x_{01})$. Also

$$0 = g(e, e, y_{01}, x_{01}) = (e, y_{01}, x_{01}) + (e, e, y_{01}x_{01} - x_{01}y_{01}) - e \circ (e, y_{01}, x_{01}).$$

The second term vanishes by hypothesis, while $(e, y_{01}, x_{01}) = -e(y_{01}x_{01}) = -y_{01}x_{01}$, by the previous calculation. Substituting this in the preceding equation we find that $e(y_{01}x_{01}) + (y_{01}x_{01})e = y_{01}x_{01}$. After cancellation we are left with $(y_{01}x_{01})e = 0$. Thus $y_{01}x_{01} \in R_{10}$. By going to the anti-isomorphic copy of R it becomes clear that $x_{10}y_{10} \in R_{01}$. Hence

$$R_{01}R_{01} \subset R_{10}, \quad \text{and} \quad R_{10}R_{10} \subset R_{01}. \quad (24)$$

Then

$$\begin{aligned} 0 &= h(x_{10}, e, y_{01}, e) = (x_{10}e - ex_{10}, y_{01}, e) \\ &\quad + (x_{10}, e, y_{01}e) - y_{01}(x_{10}, e, e) - (x_{10}, e, y_{01})e. \end{aligned}$$

The third and fourth terms of the last equation vanish and what remains is $-(x_{10}, y_{01}, e) + (x_{10}, e, y_{01}) = 0$. Again the second term of the last equation vanishes, so that $(x_{10}y_{01})e = x_{10}y_{01}$. Also $0 = g(e, x_{10}, e, y_{01}) =$

$(ex_{10}, e, y_{01}) + (e, x_{10}, ey_{01} - y_{01}e) - e(x_{10}, e, y_{01}) - (e, e, y_{01})x_{10}$. The first, third and fourth terms vanish so that $-(e, x_{10}, y_{01}) = 0$. Thence $e(x_{10}y_{01}) = x_{10}y_{01}$. From these calculations we deduce that $x_{10}y_{01} \in R_{11}$. Reversing subscripts it follows that $x_{01}y_{10} \in R_{00}$. Thus

$$R_{10}R_{01} \subset R_{11}, \quad \text{and} \quad R_{01}R_{10} \subset R_{00}. \quad (25)$$

Then

$$\begin{aligned} 0 &= h(x_{01}, e, y_{11}, e) = (x_{01}e - ex_{01}, y_{11}, e) \\ &\quad + (x_{01}, e, y_{11}e) - y_{11}(x_{01}, e, e) - (x_{01}, e, y_{11})e. \end{aligned}$$

By expansion all but the first term of the last equation vanish. What remains is $(x_{01}, y_{11}, e) = 0 = (x_{01}y_{11})e - x_{01}y_{11}$. But also $0 = g(e, e, x_{01}, y_{11}) = (e, x_{01}, y_{11}) + (e, e, x_{01}y_{11} - y_{11}x_{01}) - e \circ (e, x_{01}, y_{11})$. The second term of the last equation vanishes. The remaining equation implies that $(e, x_{01}, y_{11}) \in R_{10} + R_{01}$. Since $(x_{01}y_{11})e = x_{01}y_{11}$, we may assume that $x_{01}y_{11} = a_{11} + b_{01}$. Then $(e, x_{01}, y_{11}) = -e(x_{01}y_{11}) = -e(a_{11} + b_{01}) = -a_{11}$. Since $(R_{10} + R_{01}) \cap R_{11} = 0$, we have $a_{11} = 0$. Thus $x_{01}y_{11} = b_{01}$, so that $x_{01}y_{11} \in R_{01}$. By going to the anti-isomorphic copy of R it becomes plain that $y_{11}x_{10} \in R_{10}$. Then by reversing subscripts $y_{00}x_{01} \in R_{01}$, and $x_{10}y_{00} \in R_{10}$. Thus

$$R_{01}R_{11} \subset R_{01}, \quad R_{11}R_{10} \subset R_{10}, \quad R_{00}R_{01} \subset R_{01} \quad \text{and} \quad R_{10}R_{00} \subset R_{10}. \quad (26)$$

Then

$$\begin{aligned} 0 &= g(e, x_{11}, y_{11}, e) = (ex_{11}, y_{11}, e) \\ &\quad + (e, x_{11}, y_{11}e - ey_{11}) - e(x_{11}, y_{11}, e) - (e, y_{11}, e)x_{11}. \end{aligned}$$

The second and fourth terms vanish and what remains is $(x_{11}, y_{11}, e) = e(x_{11}, y_{11}, e)$. Also

$$\begin{aligned} 0 &= g(x_{11}, e, y_{11}, e) = (x_{11}e, y_{11}, e) \\ &\quad + (x_{11}, e, y_{11}e - ey_{11}) - x_{11}(e, y_{11}, e) - (x_{11}, y_{11}, e)e, \end{aligned}$$

so that $(x_{11}, y_{11}, e) = (x_{11}, y_{11}, e)e$. So far this establishes $(x_{11}, y_{11}, e) \in R_{11}$. But then

$$\begin{aligned} 0 &= h(x_{11}, y_{11}, e, e) = (x_{11}y_{11} - y_{11}x_{11}, e, e) \\ &\quad + (x_{11}, y_{11}, e) - e \circ (x_{11}, y_{11}, e). \end{aligned}$$

The first term vanishes by hypothesis, and what remains is $-(x_{11}, y_{11}, e) = 0$. But this implies $(x_{11}y_{11})e = x_{11}y_{11}$. Going to the anti-isomorphic copy of R it becomes clear that $(e, y_{11}, x_{11}) = 0$, so that $e(y_{11}x_{11}) = y_{11}x_{11}$. But then

$e(x_{11}y_{11}) = x_{11}y_{11}$, and $x_{11}y_{11} \in R_{11}$. By interchanging subscripts, also $x_{00}y_{00} \in R_{00}$. Therefore,

$$R_{11}R_{11} \subset R_{11}, \quad \text{and} \quad R_{00}R_{00} \subset R_{00}. \quad (27)$$

Equations (22)–(27) imply that for $i, j, k, l = 0$ or 1 , $R_{ij}R_{kl} = 0$, when $j \neq k$ and $R_{ij}R_{jl} \subset R_{il}$, with two exceptions, namely $R_{01}R_{01} \subset R_{10}$ and $R_{10}R_{10} \subset R_{01}$. Whenever we want to refer to this information, we shall call it the table of the Peirce decomposition, or simply the table.

By linearizing (iii) twice one obtains the identity

$$(x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) + (z, y, x) + (y, x, z) = 0.$$

If we let $x = x_{10}$, $y = y_{01}$ and $z = z_{11}$ in the last equation then it follows from the table of the Peirce decomposition that the last three terms vanish. What remains is $(x_{10}, y_{01}, z_{11}) + (y_{01}, z_{11}, x_{10}) + (z_{11}, x_{10}, y_{01}) = 0$. However (x_{10}, y_{01}, z_{11}) and (z_{11}, x_{10}, y_{01}) both belong to R_{11} , while $(y_{01}, z_{11}, x_{10}) \in R_{00}$. From the directness of the Peirce decomposition it must be that $(y_{01}, z_{11}, x_{10}) = 0$, and $(x_{10}, y_{01}, z_{11}) + (z_{11}, x_{10}, y_{01}) = 0$. Reversing subscripts it also follows that $(y_{10}, z_{00}, x_{01}) = 0$, and

$$(x_{01}, y_{10}, z_{00}) + (z_{00}, x_{01}, y_{10}) = 0.$$

Thus

$$(R_{01}, R_{11}, R_{10}) = 0 = (R_{10}, R_{00}, R_{01}), \quad (28)$$

and

$$(x_{10}, y_{01}, z_{11}) + (z_{11}, x_{10}, y_{01}) = 0 = (x_{01}, y_{10}, z_{00}) + (z_{00}, x_{01}, y_{10}). \quad (29)$$

Now

$$\begin{aligned} 0 &= g(a_{01}, b_{11}, c_{11}, d_{10}) = (a_{01}b_{11}, c_{11}, d_{10}) \\ &\quad + (a_{01}, b_{11}, c_{11}d_{10} - d_{10}c_{11}) - a_{01}(b_{11}, c_{11}, d_{10}) - (a_{01}, c_{11}, d_{10})b_{11}. \end{aligned}$$

Using the table of the Peirce decomposition together with (28), all but the third term vanish. Thus $a_{01}(b_{11}, c_{11}, d_{10}) = 0$. By going to the anti-isomorphic copy of R it also follows that $(d_{01}, c_{11}, b_{11})a_{10} = 0$. Hence

$$R_{01}(R_{11}, R_{11}, R_{10}) = 0 = (R_{01}, R_{11}, R_{11})R_{10}. \quad (30)$$

From

$$\begin{aligned} 0 &= f(a_{01}, b_{11}, c_{11}, d_{11})e_{10} = (a_{01}b_{11}, c_{11}, d_{11})e_{10} \\ &\quad - (a_{01}, b_{11}c_{11}, d_{11})e_{10} + (a_{01}, b_{11}, c_{11}d_{11})e_{10} \\ &\quad - [a_{01}(b_{11}, c_{11}, d_{11})]e_{10} - [(a_{01}, b_{11}, c_{11})d_{11}]e_{10}, \end{aligned}$$

it follows, using the table of the Peirce decomposition in conjunction with (30), that all but the last two terms vanish. Also since $(a_{01}, b_{11}, c_{11}) = s_{01}$, $[(a_{01}, b_{11}, c_{11})d_{11}]e_{10} = [s_{01}d_{11}]e_{10} = s_{01}[d_{11}e_{10}]$, using (28). But then $s_{01}[d_{11}e_{10}] = s_{01}t_{10} = (a_{01}, b_{11}, c_{11})t_{10} = 0$, using (30). So only one term survives from the previous equation, namely $-[a_{01}(b_{11}, c_{11}, d_{11})]e_{10} = 0$. Since $(b_{11}, c_{11}, d_{11}) \in R_{11}$, then in view of (28) we could dispense with the square brackets in the last equation. Also

$$a_{01}[(b_{11}, c_{11}, d_{11})e_{11}]f_{10} = a_{01}[(b_{11}, c_{11}, d_{11}) \cdot e_{11}f_{10}],$$

since $(b_{11}, c_{11}, d_{11}) \in R_{11}$ and we may use (30). But

$$a_{01}[(b_{11}, c_{11}, d_{11}) \cdot e_{11}f_{10}] \in R_{01}(R_{11}, R_{11}, R_{11})R_{10} = 0,$$

as we observed two equations before. We have shown

$$R_{01}(R_{11}, R_{11}, R_{11})R_{10} = R_{01}[(R_{11}, R_{11}, R_{11})R_{11}]R_{10} = 0. \quad (31)$$

From linearizing (iii) it follows that

$$0 = (e, x_{10}, x_{10}) + (x_{10}, x_{10}, e) + (x_{10}, e, x_{10}).$$

Since $x_{10}^2 \in R_{01}$ as a result of (24), the last equation becomes $x_{10}^2 + x_{10}^2 - x_{10}^2 = 0 = x_{10}^2$. A linearization of this last equation becomes $x_{10}y_{10} + y_{10}x_{10} = 0$. Thus

$$x_{10}^2 = 0 = x_{10}y_{10} + y_{10}x_{10}. \quad (32)$$

Also

$$\begin{aligned} 0 &= g(e, x_{10}, y_{10}, z_{10}) = (ex_{10}, y_{10}, z_{10}) \\ &\quad + (e, x_{10}, y_{10}z_{10} - z_{10}y_{10}) - e(x_{10}, y_{10}, z_{10}) - (e, y_{10}, z_{10})x_{10}. \end{aligned}$$

It follows from the table that the second term of the last identity vanishes. Thus we are left with $(x_{10}, y_{10}, z_{10}) + x_{10}(y_{10}z_{10}) - (y_{10}z_{10})x_{10} = 0 = (x_{10}y_{10})z_{10} - (y_{10}z_{10})x_{10}$. If we combine the last identity with (32) it becomes obvious that

$$\begin{aligned} (x_{10}y_{10})z_{10} &= (y_{10}z_{10})x_{10} = (z_{10}x_{10})y_{10} = -(y_{10}x_{10})z_{10} \\ &= -(x_{10}z_{10})y_{10} = -(z_{10}y_{10})x_{10}. \end{aligned} \quad (33)$$

By changing subscripts, we obtain from (32) and (33) that

$$x_{01}^2 = 0 = x_{01}y_{01} + y_{01}x_{01}, \quad (34)$$

and

$$\begin{aligned} (x_{01}y_{01})z_{01} &= (y_{01}z_{01})x_{01} = (z_{01}x_{01})y_{01} = -(x_{01}z_{01})y_{01} \\ &= -(z_{01}y_{01})x_{01} = -(y_{01}x_{01})z_{01}. \end{aligned} \quad (35)$$

Then

$$0 = h(b_{01}, c_{11}, a_{10}, d_{11}) = (b_{01}c_{11} - c_{11}b_{01}, a_{10}, d_{11}) \\ + (b_{01}, c_{11}, a_{10}d_{11}) - a_{10}(b_{01}, c_{11}, d_{11}) - (b_{01}, c_{11}, a_{10})d_{11}.$$

It follows from the table that all but the third term of the last equation vanish. Thus

$$R_{10}(R_{01}, R_{11}, R_{11}) = 0. \quad (36)$$

We define

$$S_{10} = \{x_{10} \in R_{10} \mid x_{10}R_{01} = 0 = R_{01}x_{10}\}$$

and

$$S_{01} = \{y_{01} \in R_{01} \mid y_{01}R_{10} = 0 = R_{10}y_{01}\}.$$

Then let $T = S_{10} + S_{01}$. For $s_{01} \in S_{01}$ we have $(s_{01}x_{11})y_{10} = s_{01}(x_{11}y_{10}) = 0$, using (28), the table of the Peirce decomposition and the definition of S_{01} . Clearly $-(y_{10}, s_{01}, x_{11}) = y_{10}(s_{01}x_{11})$. Using (29) shows that

$$-(y_{10}, s_{01}, x_{11}) = (x_{11}, y_{10}, s_{01}).$$

But $(x_{11}, y_{10}, s_{01}) = (x_{11}y_{10})s_{01} - x_{11}(y_{10}s_{01}) = 0$, using the table and the definition of S_{01} . Thus $y_{10}(s_{01}x_{11}) = 0$. Consequently $S_{01}R_{11} \subset S_{01}$. We know by definition of S_{01} that $S_{01}R_{10} = 0 = R_{10}S_{01}$. Also it follows from the table that $R_{11}S_{01} = 0 = S_{01}R_{00}$. Using (35) we find that $(s_{01}x_{01})y_{01} = (x_{01}y_{01})s_{01} = z_{10}s_{01} = 0$, as a result of (24) and the definition of S_{01} . By going to the anti-isomorphic copy of R , we see that (33) becomes

$$z_{01}(y_{01}x_{01}) = y_{01}(x_{01}z_{01}) = x_{01}(z_{01}y_{01}) \\ = -z_{01}(x_{01}y_{01}) = -y_{01}(z_{01}x_{01}) = -x_{01}(y_{01}z_{01}). \quad (37)$$

Therefore $y_{01}(s_{01}x_{01}) = s_{01}(x_{01}y_{01}) = s_{01}z_{10} = 0$. Now we have established that $s_{01}x_{01} \in S_{10}$. Hence $S_{01}R_{01} \subset S_{10}$. Then use of (34) implies that $R_{01}S_{01} \subset S_{10}$. Likewise by going to the anti-isomorphic copy of R , it becomes clear that $R_{10}S_{10} \subset S_{01}$, and $S_{10}R_{10} \subset S_{01}$. For the same reason $R_{11}S_{10} \subset S_{10}$ and $S_{10}R_{00} \subset S_{10}$ follow from the previous inclusions. We are ready to prove

LEMMA 2. *T is an ideal of R such that $t \in T$ implies $t^2 = 0$.*

Proof. We have already seen that T is an ideal of R . For every $t \in T$, $t = s_{10} + s_{01}$. But then $t^2 = s_{10}^2 + s_{10}s_{01} + s_{01}s_{10} + s_{01}^2 = 0$, using (32), (34) and the definitions of S_{10} and S_{01} . This completes the proof of the lemma.

Since R can have no proper nil ideals and e cannot be nil, we must have $T = 0$. At this point we do not need to use the full hypothesis of simplicity, as it suffices that there exist no proper nil ideal. Once this has been established, we note that because of (30) and (36) we know that

$$(R_{01}, R_{11}, R_{11}) \subset S_{01} \subset T = 0.$$

By going to the anti-isomorphic copy of R as well as by changing subscripts, we obtain

$$\begin{aligned} (R_{01}, R_{11}, R_{11}) = 0 &= (R_{11}, R_{11}, R_{10}) = (R_{10}, R_{00}, R_{00}) \\ &= (R_{00}, R_{00}, R_{01}). \end{aligned} \quad (38)$$

Now

$$\begin{aligned} 0 &= g(w_{01}, x_{11}, y_{11}, z_{11}) = (w_{01}x_{11}, y_{11}, z_{11}) \\ &\quad + (w_{01}, x_{11}, y_{11}z_{11} - z_{11}y_{11}) - w_{01}(x_{11}, y_{11}, z_{11}) - (w_{01}, y_{11}, z_{11})x_{11}. \end{aligned}$$

As a result of (38) and the table all but the third term vanish, so that $w_{01}(x_{11}, y_{11}, z_{11}) = 0$. Define A as the additive subgroup of R generated by all elements of the form (a_{11}, b_{11}, c_{11}) and $(a_{11}, b_{11}, c_{11})d_{11}$. It is well known, but can readily be verified from the Teichmüller identity that A is an ideal of the ring R_{11} , even when no identities are assumed to hold in R_{11} . From the last equation and (38) it readily follows that $R_{01}A = 0$. By going to the anti-isomorphic copy of R it then follows that $AR_{10} = 0$. Since $A \subset R_{11}$, we know from the table that $R_{00}A = 0 = AR_{00} = R_{10}A = AR_{01}$. This proves that A is an ideal of R , contained in R_{11} . If $A = R$, then $e = 1$, contrary to assumption. Thus $A = 0$, so that R_{11} must be associative. By reversing subscripts the same argument goes over except in one place. The reason why $A_{00} \neq R$ is that $e \notin A_{00}$. Thus

$$R_{11} \text{ and } R_{00} \text{ are associative subrings of } R. \quad (39)$$

At this point it becomes clear that all associators, with components from R_{ij} vanish, except possibly when at least two components belong to either R_{10} or R_{01} . In case all components lie in R_{10} , then (33) and the identity implied by changing subscripts in (37) suffice to prove the alternative law. When all three components lie in R_{01} , we use (35) and (37). Then $(x_{10}, y_{10}, z_{11}) = (x_{10}y_{10})z_{11}$, while $(y_{10}, x_{10}, z_{11}) = (y_{10}x_{10})z_{11}$. However $(x_{10}, y_{10}, z_{11}) = -(y_{10}, x_{10}, z_{11})$, for $(x_{10}y_{10})z_{11} + (y_{10}x_{10})z_{11} = 0$, using (32). Also $(x_{10}, z_{11}, y_{10}) = -x_{10}(z_{11}y_{10}) = (z_{11}y_{10})x_{10} = (z_{11}, y_{10}, x_{10})$, using (32) and the table. Besides $(y_{10}, z_{11}, x_{10}) = (z_{11}, x_{10}, y_{10})$, since x_{10} and y_{10} are interchangeable in the preceding equation. Now $0 = g(x_{10}, e, y_{10}, z_{11}) = (x_{10}e, y_{10}, z_{11}) + (x_{10}, e, y_{10}z_{11} - z_{11}y_{10}) - x_{10}(e, y_{10}, z_{11}) - (x_{10}, y_{10}, z_{11})e$.

After cancellation we see that $x_{10}(z_{11}y_{10}) = (x_{10}y_{10})z_{11}$, so that

$$-(x_{10}, z_{11}, y_{10}) = (x_{10}, y_{10}, z_{11}).$$

We may interchange x_{10} and y_{10} in the last identity to obtain

$$-(y_{10}, z_{11}, x_{10}) = (y_{10}, x_{10}, z_{11}),$$

thus establishing the alternative identities for two components in R_{10} and one component in R_{11} . By going to the anti-isomorphic copy of R as well as by interchanging subscripts one obtains the alternative identities for x_{01}, y_{01}, z_{11} , for x_{01}, y_{01}, z_{00} and for x_{10}, y_{10}, z_{00} . Also

$$\begin{aligned} (x_{10}, y_{10}, z_{01}) &= (x_{10}y_{10})z_{01} = -z_{01}(x_{10}y_{10}) = (z_{01}, x_{10}, y_{10}) \\ &= -(z_{01}, y_{10}, x_{10}) = -(y_{10}, x_{10}, z_{01}), \end{aligned}$$

using (32) and (34). From a linearization of (iii), together with the preceding equation it follows that $(x_{10}, z_{01}, y_{10}) + (y_{10}, z_{01}, x_{10}) = 0$. But

$$\begin{aligned} 0 &= g(x_{10}, z_{01}, e, y_{10}) = (x_{10}z_{01}, e, y_{10}) \\ &\quad + (x_{10}, z_{01}, ey_{10} - y_{10}e) - x_{10}(z_{01}, e, y_{10}) - (x_{10}, e, y_{10})z_{01}. \end{aligned}$$

The first and third terms of the last equation vanish and so

$$0 = (x_{10}, z_{01}, y_{10}) + (x_{10}y_{10})z_{01} = (x_{10}, z_{01}, y_{10}) + (x_{10}, y_{10}, z_{01}).$$

This suffices to establish the alternative identities for x_{10}, y_{10}, z_{01} . By changing subscripts we obtain the same for x_{01}, y_{01}, z_{10} . This completes the proof that R must be alternative.

The main remaining objective is to show that we can remove the hypothesis regarding $(e, e, R) = 0 = (e, R, e) = (R, e, e)$. So we will assume from now on that R is a ring of characteristic different from two and three, satisfying identities (i)–(iii), containing an idempotent $e \neq 1$, and containing no proper nil ideal. Since R is power-associative, we have the usual Albert decomposition of R as a direct sum of $R_1 + R_{1/2} + R_0$ [1], where $ex_1 = x_1 = x_1e$, $x_0e = 0 = ex_0$, and $ex_{1/2} + x_{1/2}e = x_{1/2}$. Using (7) and (8), with $x = e$, $y = x$, we see that $(e, x, e) = 2e(e, x, e) = 2(e, x, e)e$. If $x \in R_1$ or $x \in R_0$, then obviously $(e, x, e) = 0$. Assume that $x \in R_{1/2}$, and let $ex = x_1 + x_{1/2} + x_0$. Then $(e, e, x) = ex - e(ex) = x_1 + x_{1/2} + x_0 - x_1 - ex_{1/2} = x_0 + x_{1/2}e$. Also $(x, e, e) = (xe)e - xe = -(ex)e$, since $ex + xe = x$. But $-(ex)e = -x_1 - x_{1/2}e$, so that $(x, e, e) = -x_1 - x_{1/2}e$. From a linearization of (iii) we obtain that $(e, x, e) = -(x, e, e) - (e, e, x)$. By substituting, the last equation becomes $(e, x, e) = -x_0 - x_{1/2}e + x_1 + x_{1/2}e = x_1 - x_0$. But then $(e, x, e) = 2e(e, x, e)$ implies $x_1 - x_0 = 2x_1$, so that $x_0 = 0 = x_1$.

But now $(e, x, e) = x_1 - x_0 = 0$. We have proved

$$(e, R, e) = 0. \quad (40)$$

Then

$$\begin{aligned} 0 &= g(x_1, e, y_1, e) = (x_1 e, y_1, e) \\ &\quad + (x_1, e, y_1 e - e y_1) - x_1(e, y_1, e) - (x_1, y_1, e)e. \end{aligned}$$

The second term vanishes, as does the third, using (40). What remains reads $(x_1, y_1, e) - (x_1, y_1, e)e = 0$, so that $(x_1, y_1, e)e = (x_1, y_1, e)$. Also $g(e, x_1, y_1, e) = (e x_1, y_1, e) + (e, x_1, y_1 e - e y_1) - e(x_1, y_1, e) - (e, y_1, e)x_1$. Again the second term vanishes, as does the fourth, using (40), leaving $(x_1, y_1, e) = e(x_1, y_1, e)$. This implies that $(x_1, y_1, e) \in R_1$. Let $x_1 y_1 = z_0 + z_{1/2} + z_1$. Then

$$(x_1, y_1, e) = (x_1 y_1)e - x_1 y_1 = z_{1/2}e + z_1 - z_0 - z_{1/2} - z_1 = a_1.$$

Cancelling and multiplying through on the left by e , we see that $e(z_{1/2}e - z_{1/2}) = a_1$, so that $e(z_{1/2}e) = e z_{1/2} + a_1 = z_{1/2} - z_{1/2}e + a_1 = -z_0$, using a previous equation. Starting again with $z_{1/2}e - z_0 - z_{1/2} = a_1$, we see that $-e z_{1/2} - z_0 = a_1$. Multiplying the last equation through by e on the right we see that $-(e z_{1/2})e = a_1$. Since $(e z_{1/2})e = e(z_{1/2}e)$, using (40) we assert that $-a_1 = -z_0$. Since the Albert decomposition is direct, it follows that $a_1 = z_0 = 0$. Hence $z_{1/2}e = z_{1/2}$, and $e z_{1/2} = 0$. By going to the anti-isomorphic copy of R it is clear that $(e, x_1, y_1) = 0$. Thus $x_1 y_1 = e(x_1 y_1) = z_0 + z_{1/2} + z_1 = e z_{1/2} + z_1$. Using $z_0 = 0 = e z_{1/2}$, we find after cancelling that $z_{1/2} = 0$. At this point $x_1 y_1 = z_1$, so that R_1 is a subring of R . By reversing subscripts one can also prove R_0 to be a subring.

$$R_1 \text{ and } R_0 \text{ are subrings of } R. \quad (41)$$

Now

$$\begin{aligned} 0 &= g(y_{1/2}, e, x_1, e) = (y_{1/2}e, x_1, e) \\ &\quad + (y_{1/2}, e, x_1 e - e x_1) - y_{1/2}(e, x_1, e) - (y_{1/2}, x_1, e)e. \end{aligned}$$

Clearly the second and third terms vanish, using (40), so that $(y_{1/2}e, x_1, e) = (y_{1/2}, x_1, e)e$. Also

$$\begin{aligned} 0 &= g(e, y_{1/2}, x_1, e) = (e y_{1/2}, x_1, e) \\ &\quad + (e, y_{1/2}, x_1 e - e x_1) - e(y_{1/2}, x_1, e) - (e, x_1, e)y_{1/2}. \end{aligned}$$

The second and fourth terms vanish, leaving $(e y_{1/2}, x_1, e) = e(y_{1/2}, x_1, e)$. Combining these two calculations with $e y_{1/2} + y_{1/2}e = y_{1/2}$, it follows

that $(y_{1/2}, x_1, e) = (ey_{1/2} + y_{1/2}e, x_1, e) = e \circ (y_{1/2}, x_1, e)$. This implies $(y_{1/2}, x_1, e) \in R_{1/2}$. Let $y_{1/2}x_1 = a_1 + a_{1/2} + a_0$. Then

$$a_1 + a_{1/2}e - a_1 - a_{1/2} - a_0 = (y_{1/2}, x_1, e) = b_{1/2}.$$

Hence $-ea_{1/2} - a_0 = b_{1/2}$. Multiplying through by e on the right we see that $(ea_{1/2})e = -b_{1/2}e$. Since $b_{1/2} = -a_{1/2} + a_{1/2}e - a_0$, multiplying through on the left by e it follows that $eb_{1/2} = -ea_{1/2} + e(a_{1/2}e) = a_0 + b_{1/2} + e(a_{1/2}e)$. Using (40) we see that $e(a_{1/2}e) = (ea_{1/2})e = -b_{1/2}e$. Substituting this in the preceding equation, $eb_{1/2} = a_0 + b_{1/2} - b_{1/2}e$. This implies, by cancellation, that $a_0 = 0$. Hence $y_{1/2}x_1 = a_1 + a_{1/2}$, so that

$$R_{1/2}R_1 \subset R_1 + R_{1/2}. \quad (42)$$

Now because of (42), $z_{1/2}e \in R_1 + R_{1/2}$. But $ez_{1/2} = z_{1/2} - z_{1/2}e$, so that $ez_{1/2} \in R_1 + R_{1/2}$. Let $ez_{1/2} = q_1 + q_{1/2}$. Then $z_{1/2}e = z_{1/2} - q_1 - q_{1/2}$. Also $(e, e, z_{1/2}) = ez_{1/2} - e(ez_{1/2}) = q_1 + q_{1/2} - q_1 - eq_{1/2} = q_{1/2}e$, while

$$\begin{aligned} (z_{1/2}, e, e) &= (z_{1/2}e)e - z_{1/2}e = z_{1/2}e - q_1 - q_{1/2}e - z_{1/2} + q_1 + q_{1/2} \\ &= z_{1/2} - q_1 - q_{1/2} - q_{1/2}e - z_{1/2} + q_{1/2} = -q_1 - q_{1/2}e. \end{aligned}$$

As a result of a linearization of (iii) and (40) we have $(e, e, z_{1/2}) + (z_{1/2}, e, e) = -(e, z_{1/2}, e) = 0$. Substituting we get $q_{1/2}e - q_1 - q_{1/2}e = 0$, so that $q_1 = 0$. Thus $ez_{1/2} = q_{1/2}$. We have shown

$$ez_{1/2} \in R_{1/2}, \quad z_{1/2}e \in R_{1/2}. \quad (43)$$

But

$$\begin{aligned} 0 &= g(e, x_1, e, y_0) = (ex_1, e, y_0) \\ &\quad + (e, x_1, ey_0 - y_0e) - e(x_1, e, y_0) - (e, e, y_0)x_1. \end{aligned}$$

Clearly the second and fourth terms vanish, so that

$$(x_1, e, y_0) - e(x_1, e, y_0) = 0.$$

Consequently $x_1y_0 = e(x_1y_0)$. Then

$$\begin{aligned} 0 &= g(x_1, e, e, y_0) = (x_1e, e, y_0) \\ &\quad + (x_1, e, ey_0 - y_0e) - x_1(e, e, y_0) - (x_1, e, y_0)e, \end{aligned}$$

so that $(x_1, e, y_0) = (x_1, e, y_0)e$. Hence $(x_1y_0) = (x_1y_0)e$. This shows that $x_1y_0 \in R_1$. Similarly we may expand $0 = g(e, y_0, e, x_1)$, and $0 = g(y_0, e, e, x_1)$, to obtain that $y_0x_1 \in R_0$. But now

$$0 = h(x_1, y_0, e, e) = (x_1y_0 - y_0x_1, e, e) + (x_1, y_0, e) - e \circ (x_1, y_0, e).$$

The first term is zero by expansion, since $x_1 y_0 - y_0 x_1 \in R_0 + R_1$. Also $(x_1, y_0, e) = (x_1 y_0)e - x_1(y_0 e) = x_1 y_0 = a_1$. But then

$$a_1 - e \circ a_1 = a_1 - 2a_1 = -a_1 = 0.$$

Hence $x_1 y_0 = 0$. Similarly $y_0 x_1 = 0$. Thus

$$R_1 R_0 = 0 = R_0 R_1. \quad (44)$$

Then

$$\begin{aligned} 0 &= g(y_{1/2}, e, e, x_1) = (y_{1/2}e, e, x_1) \\ &\quad + (y_{1/2}, e, ex_1 - x_1e) - y_{1/2}(e, e, x_1) - (y_{1/2}, e, x_1)e. \end{aligned}$$

The second and third terms vanish, leaving $(y_{1/2}e, e, x_1) = (y_{1/2}, e, x_1)e$. Then

$$\begin{aligned} 0 &= g(e, y_{1/2}, e, x_1) = (ey_{1/2}, e, x_1) \\ &\quad + (e, y_{1/2}, ex_1 - x_1e) - e(y_{1/2}, e, x_1) - (e, e, x_1)y_{1/2}. \end{aligned}$$

The second and fourth terms of the last equation vanish, leaving

$$(ey_{1/2}, e, x_1) = e(y_{1/2}, e, x_1).$$

But then $(y_{1/2}, e, x_1) = (e \circ y_{1/2}, e, x_1) = e \circ (y_{1/2}, e, x_1)$, implying that $(y_{1/2}, e, x_1) \in R_{1/2}$. But $(y_{1/2}, e, x_1) = (y_{1/2}e)x_1 - y_{1/2}x_1 = -(ey_{1/2})x_1$. Thus

$$(ey_{1/2})x_1 \in R_{1/2}. \quad (45)$$

Prior to establishing (42) we noted that $(y_{1/2}, x_1, e) \in R_{1/2}$. Hence

$$\begin{aligned} 0 &= h(y_{1/2}, x_1, e, e) = (y_{1/2}x_1 - x_1y_{1/2}, e, e) \\ &\quad + (y_{1/2}, x_1, e) - e \circ (y_{1/2}, x_1, e) = (y_{1/2}x_1 - x_1y_{1/2}, e, e). \end{aligned}$$

Using (40) and a linearization of (iii) it follows that $(a, e, e) + (e, e, a) = -(e, a, e) = 0$, so that $(a, e, e) = -(e, e, a)$. Hence we get

$$(e, e, y_{1/2}x_1 - x_1y_{1/2}) = 0.$$

But then

$$\begin{aligned} 0 &= g(e, e, y_{1/2}, x_1) = (e, y_{1/2}, x_1) + (e, e, y_{1/2}x_1 - x_1y_{1/2}) \\ &\quad - e \circ (e, y_{1/2}, x_1) = (e, y_{1/2}, x_1) - e \circ (e, y_{1/2}, x_1). \end{aligned}$$

Therefore $(e, y_{1/2}, x_1) \in R_{1/2}$. Similarly $0 = g(e, e, x_1, y_{1/2})$ leads to $(e, x_1, y_{1/2}) \in R_{1/2}$. Now from $0 = h(x_1, y_{1/2}, e, e)$ it follows that

$(x_1, y_{1/2}, e) \in R_{1/2}$. Prior to (45) we proved that $(y_{1/2}, e, x_1) \in R_{1/2}$. Now a linearization of (iii) leads to

$$(a, b, c) + (b, c, a) + (c, a, b) + (a, c, b) + (c, b, a) + (b, a, c) = 0.$$

If we let $a = x_1$, $b = y_{1/2}$, and $c = e$, then five of the terms are already known to lie in $R_{1/2}$. Thus $(x_1, e, y_{1/2}) \in R_{1/2}$. We have shown

$$(x_1, e, y_{1/2}) \text{ and all permutations of this associator lie in } R_{1/2}. \quad (46)$$

Combining (45) and (46) it follows that

$$e(y_{1/2}x_1) = -(e, y_{1/2}, x_1) + (ey_{1/2})x_1 \in R_{1/2}.$$

Because of (42), $y_{1/2}x_1 = a_1 + a_{1/2}$. Then $e(y_{1/2}x_1) = ea_1 + ea_{1/2} = a_1 + ea_{1/2} \in R_{1/2}$. But $ea_{1/2} \in R_{1/2}$, because of (43), so we must have $a_1 \in R_{1/2}$ and thus $a_1 = 0$. We have proved

$$R_{1/2}R_1 \subset R_{1/2}. \quad (47)$$

By going to the anti-isomorphic copy of R , clearly (47) becomes

$$R_1R_{1/2} \subset R_{1/2}. \quad (48)$$

If x belongs to either R_0 or R_1 then $(e, e, x) = 0$, while $x \in R_{1/2}$ implies, because of (43), that $(e, e, x) \in R_{1/2}$. Thus for all $x \in R$ we have $(e, e, x) \in R_{1/2}$. But $0 = g(e, e, x, y) = (e, x, y) + (e, e, (x, y)) - e \circ (e, x, y)$. If $(e, x, y) = a_0 + a_{1/2} + a_1$, and $(e, e, (x, y)) = b_{1/2}$, then

$$0 = a_0 + a_{1/2} + a_1 + b_{1/2} - e \circ a_{1/2} - 2a_1 = a_0 + b_{1/2} - a_1.$$

Hence $b_{1/2} = a_0 = a_1 = 0$. We have proved that $(e, e, (x, y)) = 0$, and that $(e, x, y) \in R_{1/2}$. By going to the anti-isomorphic copy of R we find also that $(x, y, e) \in R_{1/2}$ and that $((x, y), e, e) = 0$. Thus

$$(e, e, (x, y)) = 0 = ((x, y), e, e). \quad (49)$$

$$(e, x, y) \in R_{1/2}, \quad (x, y, e) \in R_{1/2}. \quad (50)$$

LEMMA 3. If $B = \{x \in R_{1/2} \mid xR \subset R_{1/2}, Rx \subset R_{1/2}\}$, then $(e, e, R) \subset B$, and $(R, e, e) \subset B$.

Proof. Because of (50) we have $(e, e, x) \in R_{1/2}$. Consider $(e, e, x)y$ and $y(e, e, x)$. If $y \in R_1$ then it follows from (47) and (48) that $y(e, e, x) \in R_{1/2}$ and that $(e, e, x)y \in R_{1/2}$. By reversing subscripts in (47) and (48) it is clear that $R_0R_{1/2} \subset R_{1/2}$ and that $R_{1/2}R_0 \subset R_{1/2}$. Hence $y \in R_0$ implies that

$(e, e, x)y \in R_{1/2}$ and $y(e, e, x) \in R_{1/2}$. Without loss of generality we restrict ourselves to the case where $x, y \in R_{1/2}$. Because of (50) we have $(e, x, y) \in R_{1/2}$ and $(x, y, e) \in R_{1/2}$. Let $xy = b_0 + b_{1/2} + b_1$. Then $[ex \cdot y]_1 = [e \cdot xy]_1 = [b_1 + eb_{1/2}]_1 = b_1$, using (43). But $b_1 = [xy]_1$, so that $[ex \cdot y]_1 = [xy]_1$. Hence $[(e \cdot ex)y]_1 = [(ex)y]_1 = [xy]_1$. But then $[(e, e, x)y]_1 = 0$. Similarly $[(e, e, x)y]_0 = 0$, so that $(e, e, x)y \in R_{1/2}$. By going to the anti-isomorphic copy of R we find that $y(x, e, e) \in R_{1/2}$. But $y(x, e, e) = -y(e, e, x)$. This shows that $(e, e, R) \subset B$. This completes the proof of the lemma.

It is well known that in the Albert decomposition $x_{1/2}^2 \in R_0 + R_1$. Therefore linearization leads to $x_{1/2} \circ y_{1/2} \in R_0 + R_1$. Consequently $(e, e, x_{1/2}y_{1/2} + y_{1/2}x_{1/2}) = 0$. Because of (49) we know that

$$(e, e, x_{1/2}y_{1/2} - y_{1/2}x_{1/2}) = 0.$$

Adding and using characteristic different from two, $(e, e, x_{1/2}y_{1/2}) = 0$. Thus

$$(e, e, R_{1/2}R_{1/2}) = 0. \quad (51)$$

Let $t_{1/2} \in B$. Then for arbitrary $x, y \in R$,

$$\begin{aligned} 0 &= h(x, y, t_{1/2}, e) + h(x, y, e, t_{1/2}) \\ &= (xy - yx, t_{1/2}, e) + (x, y, t_{1/2}e) - t_{1/2}(x, y, e) - (x, y, t_{1/2})e \\ &\quad + (xy - yx, e, t_{1/2}) + (x, y, et_{1/2}) - e(x, y, t_{1/2}) - (x, y, e)t_{1/2}. \end{aligned}$$

Using (50) and the definition of B it is clear that the first, third and eighth terms of the last equation are in $R_{1/2}$. The second and sixth terms add up to $(x, y, t_{1/2})$. Hence $(x, y, t_{1/2}) + (xy - yx, e, t_{1/2}) - e \circ (x, y, t_{1/2}) \in R_{1/2}$. However $(z, e, t_{1/2}) = ze \cdot t_{1/2} - z(et_{1/2}) = c_{1/2} - z(et_{1/2})$, using the definition of B . Also $z(et_{1/2}) = zt_{1/2} - z(t_{1/2}e) = d_{1/2} - z(t_{1/2}e)$. But $-z(t_{1/2}e) = (z, t_{1/2}, e) - (zt_{1/2})e$. However (50) implies that $(z, t_{1/2}, e) \in R_{1/2}$, while (43) and the definition of B imply that $-(zt_{1/2})e \in R_{1/2}$. Putting all this together we see that $(z, e, t_{1/2}) \in R_{1/2}$, so that $(xy - yx, e, t_{1/2}) \in R_{1/2}$. But then $(x, y, t_{1/2}) - e \circ (x, y, t_{1/2}) \in R_{1/2}$. However by the nature of the element $(x, y, t_{1/2}) - e \circ (x, y, t_{1/2})$ it then follows that $(x, y, t_{1/2}) - e \circ (x, y, t_{1/2}) = 0$, and hence $(x, y, t_{1/2}) \in R_{1/2}$. By going to the anti-isomorphic copy of R then we have also $(t_{1/2}, x, y) \in R_{1/2}$. We have shown that

$$(BR)R \subset R_{1/2}, \quad \text{and} \quad R(RB) \subset R_{1/2}. \quad (52)$$

Form

$$\begin{aligned} 0 &= g(e, x_1, y_{1/2}, e) + g(x_1, e, y_{1/2}, e) \\ &= (ex_1, y_{1/2}, e) + (e, x_1, y_{1/2}e - ey_{1/2}) - e(x_1, y_{1/2}, e) - (e, y_{1/2}, e)x_1 \\ &\quad + (x_1e, y_{1/2}, e) + (x_1, e, y_{1/2}e - ey_{1/2}) - x_1(e, y_{1/2}, e) - (x_1, y_{1/2}, e)e. \end{aligned}$$

The fourth and seventh terms vanish because of (40). The first and fifth add up to $2(x_1, y_{1/2}, e) \in R_{1/2}$, using (50). But then the third and eighth add up to $-(x_1, y_{1/2}, e)$. Hence

$$(e, x_1, y_{1/2}e - ey_{1/2}) + (x_1, e, y_{1/2}e - ey_{1/2}) + (x_1, y_{1/2}, e) = 0.$$

Using a combination of (11), (3) and (4) it follows that $(x, x, (x, y)) = 0$. A linearization of this last identity shows

$$(e, e, (x, y)) + (e, x, (e, y)) + (x, e, (e, y)) = 0.$$

However $(e, e, (x, y)) = 0$, using (49). Substituting $x = x_1$, and $y = y_{1/2}$, we see that $(e, x_1, (e, y_{1/2})) + (x_1, e, (e, y_{1/2})) = 0$. Substituting this in a previous identity we find that $(x_1, y_{1/2}, e) = 0$. By going to the anti-isomorphic copy of R , also $(e, y_{1/2}, x_1) = 0$. Thus

$$(R_1, R_{1/2}, e) = 0 = (e, R_{1/2}, R_1). \quad (53)$$

Let $x, y \in R_{1/2}$. In the proof of Lemma 3 we were able to show that $[(ex)y]_1 = [xy]_1$. Hence $[(xe)y]_1 = 0$. Similarly $[(xe)y]_0 = 0$. By going to the anti-isomorphic copy of R we also know that $[x(ey)]_1 = 0 = [x(ey)]_0$. Thus $(x, e, y) \in R_{1/2}$. Observe that

$$\begin{aligned} 0 &= h(x_1, t_{1/2}, y_{1/2}, e) + h(x_1, t_{1/2}, e, y_{1/2}) \\ &= (x_1 t_{1/2} - t_{1/2} x_1, y_{1/2}, e) + (x_1, t_{1/2}, y_{1/2} e) - y_{1/2} (x_1, t_{1/2}, e) \\ &\quad - (x_1, t_{1/2}, y_{1/2}) e + (x_1 t_{1/2} - t_{1/2} x_1, e, y_{1/2}) + (x_1, t_{1/2}, e y_{1/2}) \\ &\quad - e (x_1, t_{1/2}, y_{1/2}) - (x_1, t_{1/2}, e) y_{1/2}. \end{aligned}$$

Since $ey_{1/2} + y_{1/2}e = y_{1/2}$, the second and sixth terms add up to $(x_1, t_{1/2}, y_{1/2})$. The third and eighth terms vanish as a result of (53). The first term lies in $R_{1/2}$ as a result of (50). From the observation made after (53), the fifth term also lies in $R_{1/2}$. Thus $(x_1, t_{1/2}, y_{1/2}) - e \circ (x_1, t_{1/2}, y_{1/2}) = q_{1/2} \in R_{1/2}$. Since the left hand side has $R_{1/2}$ component equal to zero, it follows that $q_{1/2} = 0$. But then $(x_1, t_{1/2}, y_{1/2}) \in R_{1/2}$. Similarly, if we go to the anti-isomorphic copy of R , also $(y_{1/2}, t_{1/2}, x_1) \in R_{1/2}$. Thus

$$(R_1, B, R_{1/2}) \subset R_{1/2}, \quad \text{and} \quad (R_{1/2}, B, R_1) \subset R_{1/2}. \quad (54)$$

Now $R_1(BR_{1/2}) \subset R_1R_{1/2} \subset R_{1/2}$, using the definition of B and (48). Then (54) implies $(R_1B)R_{1/2} \subset R_{1/2}$. Since $R_1B \subset R_{1/2}$, it is clear from (47) and its counterpart when interchanging subscripts that $(R_1B)R_1 \subset R_{1/2}$ and

$(R_1B)R_0 \subset R_{1/2}$. Thus $(R_1B)R \subset R_{1/2}$. By going to the anti-isomorphic copy of R it becomes clear that $R(BR_1) \subset R_{1/2}$. Since $(BR_1)R \subset R_{1/2}$, and $R(R_1B) \subset R_{1/2}$ was proved in (52) it now follows that

$$R_1B \subset B, \quad \text{and} \quad BR_1 \subset B. \quad (55)$$

By interchanging subscripts in (55) it also follows that

$$BR_0 \subset B, \quad \text{and} \quad R_0B \subset B. \quad (56)$$

From (52) it follows that $(t_{1/2}x_{1/2})y_{1/2} \in R_{1/2}$. Also

$$t_{1/2}x_{1/2} + x_{1/2}t_{1/2} \in R_0 + R_1,$$

while from the definition of B , $t_{1/2}x_{1/2} + x_{1/2}t_{1/2} \in R_{1/2}$. Since the Albert decomposition is direct it follows that $t_{1/2}x_{1/2} + x_{1/2}t_{1/2} = 0$. Thus $(x_{1/2}t_{1/2})y_{1/2} = -(t_{1/2}x_{1/2})y_{1/2} \in R_{1/2}$. Hence $(R_{1/2}B)R_{1/2} \subset R_{1/2}$. By going to the anti-isomorphic copy of R , also $R_{1/2}(BR_{1/2}) \subset R_{1/2}$. Now $(R_{1/2}B)R_1 \subset R_{1/2}$ follows from (47), and similarly $(R_{1/2}B)R_0 \subset R_{1/2}$. Thus $(R_{1/2}B)R \subset R_{1/2}$. Combining this with (52) we see that $R_{1/2}B \subset B$. By going to the anti-isomorphic copy of R we also obtain $BR_{1/2} \subset B$. So

$$R_{1/2}B \subset B, \quad \text{and} \quad BR_{1/2} \subset B. \quad (57)$$

Combining (55)–(57) we see that B is an ideal of R . Also $t_{1/2}^2 \in (R_0 + R_1) \cap R_{1/2}$, so $t_{1/2}^2 = 0$. Thus B is nil. Since $B \neq R$, $B = 0$. By Lemma 3, $(e, e, R) \subset B$, and $(R, e, e) \subset B$, so $(e, e, R) = 0 = (R, e, e)$. Combined with (40) this means R has a Peirce decomposition relative to e . We have proved

THEOREM 2. *A ring of characteristic different from two and three, satisfying (i)–(iii), which has no proper nil ideal, and contains an idempotent e , has a Peirce decomposition relative to e .*

Earlier in Part II we dealt with this situation under the added assumption of simplicity. Then we can prove the ring to be alternative, once we have a Peirce decomposition. Consequently

THEOREM 3. *A simple ring of characteristic different from two and three, satisfying (i)–(iii), which contains an idempotent $e \neq 1$, must be alternative, and hence either a Cayley vector-matrix algebra or associative.*

It is possible to extend these results to semi-simple rings by the usual methods that have been worked out for alternative rings, but we omit the details.

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